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Operational Phase-Space Probability Distribution in Quantum Communication Theory

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Operational phase-space probability distributions are useful tools for describing quantum mechanical systems, including quantum communication and quantum information processing systems. It is shown that quantum communication channels with Gaussian noise and quantum teleportation of continuous variables are described by operational phase-space probability distributions. The relation of operational phase-space probability distribution to the extended phase-space formalism proposed by Chountasis and Vourdas is discussed.

KEY WORDS: operational phase-space probability; quantum communication channel; quantum teleportation; Wigner and Weyl functions.

1. INTRODUCTION

The phase-space representations of quantum states of a physical system are very useful for investigating fundamental problems in quantum mechanics and quantum optics (Hillery *et al.*, 1984; Kim and Noz, 1986; Leonhardt, 1997). There are various kinds of phase-space quasiprobability distributions which describe a quantum state of a physical system. The Glauber–Sudarshan P-function (Glauber, 1963a,b; Sudarshan, 1993) is a diagonal representation with respect to the Glauber coherent states. The Husimi Q-function (Husimi, 1940; Kano, 1965) is closely related to the heterodyne detection or the simultaneous measurement of position and momentum (Leonhardt, 1997). The Wigner function (Hillery *et al.*, 1984; Wigner, 1932) gives the correct position and momentum marginal probability distributions and is related to the homodyne tomography (Leonhardt, 1997). The Glauber–Sudarshan P-function becomes singular for a nonclassical state such as the Fock state. These functions are special cases of the generalized phase-space functions (Agarwal and Wolf, 1970a,b,c; Cahill and Glauber, 1969a,b), which are called the s-ordered phase-space functions. The variety of the phase-space functions is due to the noncommutativity of quantum mechanical operators.

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When the effects of a measurement apparatus, which interacts with a physical system to obtain some information, are taken into account, the operational phase-space probability distributions (Ban, 1997; Bužek *et al.*, 1995; Wódkiewicz, 1986) are derived. The operational phase-space probability distribution reduces to the Husimi Q-function if a measurement apparatus is prepared in a vacuum state. There are other possibilities of introducing phase-space formulation of quantum mechanical systems. In fact, Chountasis and Vourdas (Chountasis and Vourdas, 1998a,b) have considered the absolute square of the Wigner and Weyl functions and they have proposed the extended phase-space formulation of quantum mechanical systems.

In this paper, we give several properties of the operational phase-space probability distributions and show that the operational phase-space probability distribution is a useful tool for investigating quantum communication systems. In Section 2, we briefly review the basic properties of the operational phase-space probability distributions. In Section 3, we show that an output state of a quantum communication channel with Gaussian noise is described by the operational phase-space probability distribution. In Section 4, we find that the quantum teleportation of continuous variables via a two-mode squeezed-vacuum state is investigated by the operational phase-space probability distribution. In Section 5, we discuss the relation between the operational phase-space probability distribution and the extended phase-space formalism by Chountasis and Vourdas. In Section 6, a summary is given.

2. OPERATIONAL PHASE-SPACE PROBABILITY DISTRIBUTION

This section briefly reviews the basic properties of the operational phase-space probability distributions (Ban, 1997; Bužek *et al.*, 1995; Wódkiewicz, 1986). The operational phase-space probability distribution $\mathcal{W}(r, k; \hat{\rho}, \hat{\sigma})$ of a quantum state $\hat{\rho}$ of a physical system is given by

$$\mathcal{W}(r, k; \hat{\rho}, \hat{\sigma}) = \frac{1}{2\pi} \mathrm{Tr}[\hat{\rho}\hat{D}(r, k)\hat{\sigma}\hat{D}^{\dagger}(r, k)], \tag{2.1}$$

where $\hat{D}(r, k)$ is the displacement operator,

$$\hat{D}(r, k) = \exp[i(k\hat{x} - r\hat{p})] = \exp(\mu\hat{a}^{\dagger} - \mu^{*}\hat{a}), \quad \mu = (r + ik)/\sqrt{2}, \tag{2.2}$$

and the statistical operator $\hat{\sigma}$ represents a quantum state of a quantum ruler (Bužek *et al.*, 1995) or a reference system (Ban, 1997), which includes the effects of a measurement apparatus. In particular, if the quantum ruler is in a vacuum state $\hat{\sigma} = |0\rangle\langle 0|$, the operational phase-space probability distribution reduces to the Husimi Q-function. It is easy to see that the operational phase-space probability distribution satisfies the symmetric relation $\mathcal{W}(-r, -k; \hat{\rho}, \hat{\sigma}) = \mathcal{W}(r, k; \hat{\sigma}, \hat{\rho})$.

In some cases, the operational phase-space probability distribution is derived by considering the simultaneous measurement of position and momentum of the physical system (Ban, 1997).

The operational phase-space probability distribution can be expressed as the convolution of the s-ordered quasiprobability and $(-s)$-ordered quasiprobability (Cahill and Glauber, 1969a,b). In particular, we obtain

$$\begin{aligned}\mathcal{W}(r, k; \hat{\rho}, \hat{\sigma}) &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\ P(x + r, p + k; \hat{\rho}) Q(x, p; \hat{\sigma}) \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\ Q(x + r, p + k; \hat{\rho}) P(x, p; \hat{\sigma}) \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\ W(x + r, p + k; \hat{\rho}) W(x, p; \hat{\sigma}), \end{aligned} \tag{2.3}$$

where $P(x, p; \hat{\rho})$, $Q(x, p; \hat{\rho})$, and $W(x, p; \hat{\rho})$ are the Glauber–Sudarshan P-function, the Huisimi Q-function, and the Wigner function of the quantum state $\hat{\rho}$ respectively. Since following relations are satisfied

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \hat{D}(r, k) \hat{\sigma} \hat{D}^{\dagger}(r, k) = \int_{-\infty}^{\infty} dx \langle x - r | \hat{\sigma} | x - r \rangle | x \rangle \langle x |, \tag{2.4}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dr \hat{D}(r, k) \hat{\sigma} \hat{D}^{\dagger}(r, k) = \int_{-\infty}^{\infty} dp \langle p - k | \hat{\sigma} | p - k \rangle | p \rangle \langle p |, \tag{2.5}$$

the operational phase-space probability distribution has the marginal distributions given by

$$\mathcal{W}(r; \hat{\rho}, \hat{\sigma}) = \int_{-\infty}^{\infty} dk \mathcal{W}(r, k; \hat{\rho}, \hat{\sigma}) = \int_{-\infty}^{\infty} dx f(x - r) \langle x | \hat{\rho} | x \rangle, \tag{2.6}$$

$$\mathcal{W}(k; \hat{\rho}, \hat{\sigma}) = \int_{-\infty}^{\infty} dr \mathcal{W}(r, k; \hat{\rho}, \hat{\sigma}) = \int_{-\infty}^{\infty} dp f(k - p) \langle p | \hat{\rho} | p \rangle, \tag{2.7}$$

where $f(x) = \langle x | \hat{\sigma} | x \rangle$ and $g(p) = \langle p | \hat{\sigma} | p \rangle$ correspond to the filter functions of the measurement apparatus.

The operational phase-space probability distribution $\mathcal{W}(r, k; \hat{\rho}, \hat{\sigma})$ given by Eq. (2.1) is expressed as

$$\mathcal{W}(r, k; \hat{\rho}, \hat{\sigma}) = \text{ Tr } [\hat{\rho} \hat{\mathcal{M}}(r, k)], \tag{2.8}$$

where the operator $\hat{\mathcal{M}}(r, k) = (2\pi)^{-1} \hat{D}(r, k) \hat{\sigma} \hat{D}^{\dagger}(r, k)$ is a positive operator-valued measure defined on the Hilbert space $\mathcal{H}$ of the system, which satisfies

$$\hat{\mathcal{M}}(r, k) \geq 0, \qquad \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\ \hat{\mathcal{M}}(r, k) = \hat{1}. \tag{2.9}$$

Note that the operator $\hat{\mathcal{M}}(r, k)$ is not a projector. Let us introduce an auxiliary Hilbert space $\mathcal{H}_{\rm a}$ and define a statistical operator by the relation,

$$\hat{\sigma}_{\rm a} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |x\rangle_{\rm a}[\langle y|\hat{\sigma}|x\rangle]_{\rm a}\langle y|. \tag{2.10}$$

We denote all of the quantities in the auxiliary Hilber space $\mathcal{H}_{\rm a}$ by adding the subscript "a." We further introduce a vector belonging to the tensor product Hilbert space $\mathcal{H} \otimes \mathcal{H}_{\rm a}$,

$$|\psi(r, k)\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx |x + r\rangle \otimes |x\rangle_{\rm a} \exp(ikx). \tag{2.11}$$

Then we find that the operational phase-space probability distribution $\mathcal{W}(r, k; \hat{\rho}, \hat{\sigma})$ can be expressed as (Ban, 1999)

$$\mathcal{W}(r, k; \hat{\rho}, \hat{\sigma}) = \text{ Tr Tr}_{\rm a}[(\hat{\rho} \otimes \hat{\sigma}_{\rm a})\hat{\mathcal{N}}(r, k)], \tag{2.12}$$

where the operator $\hat{\mathcal{N}}(r, k) = |\psi(r, k)\rangle\langle\psi(r, k)|$ is a projection-valued measure of continuous spectrum,

$$\hat{\mathcal{N}}(r, k) \geq 0, \qquad \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, \hat{\mathcal{N}}(r, k) = \hat{1} \otimes \hat{1}_{\rm a}, \tag{2.13}$$

$$\hat{\mathcal{N}}(r, k)\hat{\mathcal{N}}(r', k') = \delta(r - r')\delta(k - k')\hat{\mathcal{N}}(r, k). \tag{2.14}$$

Therefore the projection-valued measure $\hat{\mathcal{N}}(r, k)$ is the Naimark extension of the positive operator-valued measure $\hat{\mathcal{M}}(r, k)$ and the statistical operator $\hat{\sigma}_{\rm a}$ is the Naimark state (Holevo, 1982). Since the state vector $|\psi(r, k)\rangle$ is the simultaneous eigenstate of $\hat{x} \otimes \hat{1}_{\rm a} - \hat{1} \otimes \hat{x}_{\rm a}$ and $\hat{p} \otimes \hat{1}_{\rm a} + \hat{1} \otimes \hat{p}_{\rm a}$, the projection-valued measure $\hat{\mathcal{N}}(r, k)$ describes the quantum measurement process of these quantities, which is implemented by the heterodyne detection (Leonhardt, 1997).

Before concluding this section, we consider the quantum measurement of position and momentum of a physical system. Let $\hat{\rho}_{\rm a}$ and $\hat{\rho}_{\rm b}$ be statistical operators of initial quantum states of two measurement apparatus, which respectively measure the position and momentum of the physical system. Suppose that the interaction Hamiltonian between the physical system and the measurement apparatus is given by (Ban, 1997; Braunstein *et al.*, 1991)

$$H_{\rm int} = g(\hat{x} \otimes \hat{p}_{\rm a} \otimes \hat{1}_{\rm b} + \hat{p} \otimes \hat{1}_{\rm a} \otimes \hat{p}_{\rm b}). \tag{2.15}$$

After the interaction, we read the values exhibited by the two measurement apparatus. Then the joint probability distribution that the position and momentum of the physical system take the values r and k is given by the operational phase-space probability distribution $\mathcal{W}(r, k; \hat{\rho}, \hat{\sigma})$ (Ban, 1997). In this case, the quantum ruler state is given by

$$\hat{\sigma} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \times \hat{T}(x', y')\hat{T}(x, y)_{\mathrm{a}}\langle -x|\hat{\rho}_{\mathrm{a}}| - x'\rangle_{\mathrm{ab}}\langle -y|\hat{\rho}_{\mathrm{b}}| - y'\rangle_{\mathrm{b}}, \tag{2.16}$$

where the operator $\hat{T}(x, y)$ is the Fourier transformation of the displacement operator $\hat{D}(x, y)$,

$$\hat{T}(x, u) = \frac{1}{2\pi}\int_{-\infty} du \int_{-\infty}^{\infty} dv\ \hat{D}(v, u) \exp[-i(ux - vy)]. \tag{2.17}$$

Therefore we have seen that the operational phase-space probability distribution describes the simultaneous measurement of position and momentum of the system.

3. COMMUNICATION CHANNELS WITH GAUSSIAN NOISE

In this section, we show that the operational phase-space probability distribution characterizes an output state of a quantum communication channel with Gaussian noise. Suppose a quantum communication channel under the influence of Gaussian noise. When an input quantum state of a signal for this channel is given by the statistical operator $\hat{\rho}_{\mathrm{in}}$, the output quantum state of the signal (Hall, 1994) is described by the statistical operator,

$$\hat{\rho}_{\mathrm{out}} = \int d^2\beta P(\beta)\hat{D}^{\dagger}(\beta)\hat{\rho}_{\mathrm{in}}\hat{D}(\beta), \tag{3.1}$$

where $\beta = \beta_r + i\beta_i$, $d^2\beta = d\beta_r d\beta_i$, and $\hat{D}(\beta) = \hat{D}(x, p)$ is the displacement operator with $\beta = (x + ip)/\sqrt{2}$. In this equation, $P(\beta)$ represents the probability distribution of the complex amplitude β, which satisfies

$$P(\beta) \geq 0, \qquad \int d^2\beta P(\beta) = 1. \tag{3.2}$$

If the Gaussian noise is caused by the environmental system in the thermal equilibrium state, the probability distribution $P(\beta)$ is given by (Vourdas, 1988)

$$P(\beta) = \frac{1}{\pi\bar{n}} \exp\left(-\frac{|\beta|^2}{\bar{n}}\right), \tag{3.3}$$

where $\bar{n}$ is the average value of the photon number of the thermal noise.

We now calculate the Husimi Q-function of the output quantum state $\hat{\rho}_{\text{out}}$ of the signal. We obtain from Eq. (3.1)

$$
\begin{aligned}
Q(\alpha;\hat{\rho}_{\text{out}}) &= \frac{1}{\pi}\langle\alpha|\hat{\rho}_{\text{out}}|\alpha\rangle \\
&= \frac{1}{\pi}\int d^2\beta\, P(\beta)\langle\alpha|\hat{D}^{\dagger}(\beta)\hat{\rho}_{\text{in}}\hat{D}(\beta)|\alpha\rangle \\
&= \frac{1}{\pi}\,\text{Tr}\left[\hat{\rho}_{\text{in}}\int d^2\beta\, P(\beta)\hat{D}(\beta)|\alpha\rangle\langle\alpha|\hat{D}^{\dagger}(\beta)\right] \\
&= \frac{1}{\pi}\,\text{Tr}\left[\hat{\rho}_{\text{in}}\hat{D}(\alpha)\left(\int d^2\beta\, P(\beta)|\beta\rangle\langle\beta|\right)\hat{D}^{\dagger}(\alpha)\right] \\
&= \frac{1}{\pi}\,\text{Tr}[\hat{\rho}_{\text{in}}\hat{D}(\alpha)\hat{\sigma}_P\hat{D}^{\dagger}(\alpha)], \qquad (3.4)
\end{aligned}
$$

with

$$\hat{\sigma}_P = \int d^2\beta\, P(\beta)|\beta\rangle\langle\beta|, \qquad (3.5)$$

which is a statistical operator having a nonnegative Glauber–Sudarshan P-function. When $P(\beta)$ is given by Eq. (3.3), $\hat{\sigma}_P$ becomes the thermal equilibrium state,

$$\hat{\sigma}_P = \frac{1}{1+\bar{n}}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n |n\rangle\langle n|. \qquad (3.6)$$

By using position and momentum variables, x and p, the Husimi Q-function is expressed as

$$Q(x,p;\hat{\rho}_{\text{out}}) = \frac{1}{2\pi}\,\text{Tr}[\hat{\rho}_{\text{in}}\hat{D}(x,p)\hat{\sigma}_P\hat{D}^{\dagger}(x,p)] = \mathcal{W}(x,p;\hat{\rho}_{\text{in}},\hat{\sigma}_P), \qquad (3.7)$$

where we have taken account into the fact that $\alpha = (x+ip)/\sqrt{2}$. Thus the Husimi Q-function of the output state $\hat{\rho}_{\text{out}}$ of the quantum communication channel with the Gaussian noise is nothing but the operational phase-space probability distribution. Note that q quantum state $\hat{\rho}$ is expressed in terms of the Husimi Q-function $Q(\alpha;\hat{\rho})$ (Agarwal and Wolf, 1970a,b,c)

$$\hat{\rho} = \frac{1}{\pi}\int d^2\alpha\int d^2\beta\; Q(\alpha;\hat{\rho})\,\exp\left(\frac{1}{2}|\beta|^2 - \beta^*\alpha + \beta\alpha^*\right)\hat{D}(\beta). \qquad (3.8)$$

Thus it is found from Eqs. (3.4) and (3.8) that the output state of the quantum communication channel is completely determined by the operational phase-space probability distribution.

The operational phase-space probability distribution of the output quantum state with a quantum ruler state $\hat{\sigma}$ is calculated as follows:

$$\begin{aligned}\mathcal{W}(\alpha;\hat{\rho}_{\text{out}},\hat{\sigma}) &= \frac{1}{\pi}\,\text{Tr}[\hat{\rho}_{\text{out}}\hat{D}(\alpha)\hat{\sigma}\hat{D}^{\dagger}(\alpha)] \\ &= \frac{1}{2\pi}\int d^2\beta\,P(\beta)\,\text{Tr}[\hat{\rho}_{\text{in}}\hat{D}(\alpha+\beta)\hat{\sigma}\hat{D}^{\dagger}(\alpha+\beta)] \\ &= \int d^2\beta\,P(\beta-\alpha)\mathcal{W}(\beta;\hat{\rho}_{\text{in}},\hat{\sigma}),\end{aligned} \tag{3.9}$$

with $\alpha = (x+ip)/\sqrt{2}$ and $\beta = (x'+ip')/\sqrt{2}$. Thus the quantum communication channel with the Gaussian noise transforms the operational phase-space probability distribution $\mathcal{W}(\alpha;\hat{\rho}_{\text{in}},\hat{\sigma})$ of the input state into the convolution of $\mathcal{W}(\alpha;\hat{\rho}_{\text{in}},\hat{\sigma})$ with the noise distribution $P(\alpha)$. The result of this section indicates that the operational phase-space probability distribution describes the quantum communication system in which the quantum-state signal is sent through the channel with the Gaussian noise.

4. QUANTUM TELEPORTATION OF CONTINUOUS VARIABLES

4.1. Quantum Teleportation

The quantum teleportation of continuous variables can transmit any single mode optical state from Alice to Bob who share, in advance, a two-mode squeezed-vacuum state, by sending from Alice to Bob the outcomes of a simultaneous measurement of position and momentum performed by her (Braunstein and Kimble, 1998; Milburn and Braunstein, 1999). The two-mode squeezed-vacuum state $|\Psi_{\text{SV}}^{\text{AB}}\rangle$ shared by Alice and Bob is given by

$$\begin{aligned}\left|\Psi_{\text{AV}}^{\text{AB}}\right\rangle &= \exp[r(\hat{a}^{\dagger}\hat{b}^{\dagger} - ab)]|0^{\text{A}}\rangle\otimes|0^{\text{B}}\rangle \\ &= \sqrt{1-\lambda^2}\sum_{n=0}^{\infty}\lambda^n|n^{\text{A}}\rangle\otimes|n^{\text{B}}\rangle,\end{aligned} \tag{4.1}$$

where $\lambda = \tanh r$ and $(\hat{a}, \hat{a}^{\dagger})$ and $(\hat{b}, \hat{b}^{\dagger})$ are annihilation and creation operators of the mode A and B, and $|n^{\text{A}}\rangle$ and $|n^{\text{B}}\rangle$ are their number eigenstates. The mode A is assigned for Alice and the mode B for Bob.

Suppose that Alice has an unknown quantum state $|\psi^{\text{C}}\rangle$ to be teleported to Bob, where a quantum state $|\psi^{\text{C}}\rangle$ is expanded in terms of position eigenstates $|x^{\text{C}}\rangle$ as

$$|\psi^{\text{C}}\rangle = \int_{-\infty}^{\infty} dx\ \psi(x)|x^{\text{C}}\rangle. \tag{4.2}$$

Then the quantum state of the total system is $|\psi^{\mathrm{C}}\rangle \otimes |\Psi_{\mathrm{SV}}^{\mathrm{AB}}\rangle$. Note that the mode A and C are at Alice's side and the mode B at Bob's side. To teleport the unknown quantum state $|\psi^{\mathrm{C}}\rangle$ to Bob, Alice performs the simultaneous measurement of position and momentum of the mode A and C that is described the projection operator $\hat{X}^{\mathrm{AC}}(x, p) = |\Phi^{\mathrm{AC}}(x, p)\rangle\langle\Phi^{\mathrm{AC}}(x, p)|$, where $|\Phi^{\mathrm{AC}}(x, p)\rangle$ is the simultaneous eigenstate of $\hat{x}^{\mathrm{C}} - \hat{x}^{\mathrm{A}}$ and $\hat{p}^{\mathrm{C}} + \hat{p}^{\mathrm{A}}$ with eigenvalues x and p,

$$|\Phi^{\mathrm{AC}}(x, p)\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy |x^{\mathrm{C}} + y^{\mathrm{C}}\rangle \otimes |y^{\mathrm{A}}\rangle \, \exp(ipy). \tag{4.3}$$

After Alice have obtained the measurement outcome (x, p), the quantum state of the total system becomes $(\hat{X}^{\mathrm{AC}} \otimes \hat{I}^{\mathrm{B}})(|\Psi_{\mathrm{SV}}^{\mathrm{AB}}\rangle \otimes |\psi^{\mathrm{C}}\rangle)/\sqrt{P(x, p)}$, where $P(x, p)$ is the normalization constant which represents the probability of the measurement outcome (x, p).

Alice sends the measurement outcome (x, p) to Bob by means of some classical communication channel. After receiving the measurement outcome, the quantum state of the mode B held by Bob becomes

$$|\psi^{\mathrm{B}}(x, p)\rangle = \frac{|\tilde{\psi}^{\mathrm{B}}(x, p)\rangle}{\sqrt{\langle\tilde{\psi}^{\mathrm{B}}(x, p)|\tilde{\psi}^{\mathrm{B}}(x, p)\rangle}}, \tag{4.4}$$

where the unnormalized state $|\tilde{\psi}^{\mathrm{B}}(x, p)\rangle$ is given by

$$|\tilde{\psi}^{\mathrm{B}}(x, p)\rangle = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \, G_{\lambda}(z, y)\psi(x + y) \, e^{-ipy} |z^{\mathrm{B}}\rangle. \tag{4.5}$$

The kernel function $G_{\lambda}(z, y)$ is given by

$$\begin{aligned} G_{\lambda}(z, y) &= \frac{1}{\pi\sqrt{2}} \exp\left[-\frac{1}{4}\left(\frac{1+\lambda}{1-\lambda}\right)(z-y)^2 - \frac{1}{4}\left(\frac{1-\lambda}{1+\lambda}\right)(z+y)^2\right] \\ &= \frac{1}{\pi\sqrt{2}} \exp\left[-\frac{1}{4} e^{2r}(z-y)^2 - \frac{1}{4} e^{-2r}(z+y)^2\right], \end{aligned} \tag{4.6}$$

which satisfies the relations

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \, G_{\lambda}(z, y) = \sqrt{2}, \qquad \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz [G_{\lambda}(z, y)]^2 = \frac{1}{2\pi}. \tag{4.7}$$

Then Bob applies the unitary operators $e^{ip\hat{x}^{\mathrm{B}}}$ and $e^{-ixp^{\mathrm{B}}}$ and he finally obtained the quantum state $e^{-ixp^{\mathrm{B}}} e^{ip\hat{x}^{\mathrm{B}}} |\psi(x, p)\rangle$. It is easy to see that in the strong squeezing limit ($\lambda \to 1$ or $r \to \infty$), Bob's quantum state is identical with the unknown quantum state that Alice wanted to send, that is,

$$\lim_{\lambda \to 1} e^{-ixp^{\mathrm{B}}} \, e^{ip\hat{x}^{\mathrm{B}}} |\psi^{\mathrm{B}}(x, p)\rangle = \int_{-\infty}^{\infty} dx \, \psi(x) |x^{\mathrm{B}}\rangle = |\psi^{\mathrm{B}}\rangle. \tag{4.8}$$

For the finite strength of the squeezing, the teleported quantum state $e^{-ix\hat{p}^{\mathrm{B}}}e^{ip\hat{x}^{\mathrm{B}}}|\psi^{\mathrm{B}}(x,p)\rangle$ is not equal to the original quantum state $|\psi^{\mathrm{B}}\rangle$ because of the incompleteness of the quantum entanglement of the two-mode squeezed-vaccum state.

The difference between two quantum states, e.g. $|\psi_1\rangle$ and $|\psi_2\rangle$, is measured by the fidelity $F = |\langle\psi_1 \mid \psi_2\rangle|^2$. Since Bob obtained the quantum state $e^{-ix\hat{p}} e^{ip\hat{x}} \mid \psi(x,p)\rangle$ with probability $\langle\tilde{\psi}(x,p) \mid \tilde{\psi}(x,p)\rangle$, the average value of the fidelity F is calculated as

$$\begin{aligned} F &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \langle\tilde{\psi}(x,p) \mid \tilde{\psi}(x,p)\rangle |\langle\psi|e^{-ix\hat{p}}\, e^{ip\hat{x}}|\psi(x,p)\rangle|^2 \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp |\langle\psi|e^{-ix\hat{p}}\, e^{ip\hat{x}}|\tilde{\psi}(x,p)\rangle|^2, \end{aligned} \tag{4.9}$$

where the superscript “B” has been omitted since all the quantities are for Bob and there is no confusion. To proceed further, we rewrite Eq. (4.5) into

$$|\tilde{\psi}(x,p)\rangle = \hat{\mathcal{T}}_\lambda\, e^{-ip\hat{x}}\, e^{ix\hat{p}}|\psi\rangle, \tag{4.10}$$

where the operator $\hat{\mathcal{T}}_\lambda$ is given by

$$\hat{\mathcal{T}}_\lambda = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz |z\rangle G_\lambda(z,y)\langle y|, \tag{4.11}$$

which is expressed in terms of the Fock states,

$$\hat{\mathcal{T}}_\lambda = \sqrt{\frac{1-\lambda^2}{2\pi}} \sum_{n=0}^{\infty} \lambda^n |n\rangle\langle n|. \tag{4.12}$$

The operator $\hat{D}(x,p)\hat{\mathcal{T}}_\lambda \hat{D}^\dagger(x,p)$ is called the transfer operator (Hofmann *et al.*, 2000). Substituting Eq. (4.10) into Eq. (4.9), we obtain the average value of the fidelity in the continuous variables teleportation,

$$\begin{aligned} F &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp |\langle\psi|\hat{D}(x,p)\hat{\mathcal{T}}_\lambda \hat{D}^\dagger(x,p)|\psi\rangle|^2 \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp |\langle\psi|\hat{D}^\dagger(x,p)\hat{\mathcal{T}}_\lambda \hat{D}(x,p)|\psi\rangle|^2. \end{aligned} \tag{4.13}$$

For example, when $|\psi\rangle$ is the Glauber coherent state $|\alpha\rangle$, the average value of the fidelity becomes $F = (1+\lambda)/2$ (Hofmann *et al.*, 2000).

4.2. Operational Phase-Space Probability Distribution

We consider the operational phase-space probability distribution in the continuous variables teleportation. For given measurement outcome (x,p), after applying the unitary transformation $\hat{D}(x,p)$, the teleported quantum state $\hat{\rho}_{\mathrm{out}}(x,p)$

obtained by Bob can be expressed as

$$\hat{\rho}_{\text{out}}(x, p) = \frac{\hat{A}(x, p)\hat{\rho}_{\text{in}}\hat{A}^\dagger(x, p)}{\text{Tr}[\hat{A}^\dagger(x, p)\hat{A}(x, p)\hat{\rho}_{\text{in}}]}, \tag{4.14}$$

where the statistical operator $\hat{\rho}_{\text{in}}$ represents the original quantum state and the operator $\hat{A}(x, p)$ is given by

$$\hat{A}(x, p) = \hat{D}(x, p)\hat{A}\hat{D}^\dagger(x, p), \qquad \hat{A} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, G(x, y)|x\rangle\langle y|, \tag{4.15}$$

which satisfies the normalization condition

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \hat{A}^\dagger(x, p)\hat{A}(x, p) = \hat{1}. \tag{4.16}$$

Note that Eq. (4.14) is valid for both pure and mixed quantum state $\hat{\rho}_{\text{in}}$. Since the measurement outcome (x, p) is obtained with probability

$$P(x, p) = \text{Tr}[\hat{A}^\dagger(x, p)\hat{A}(x, p)\hat{\rho}_{\text{in}}], \tag{4.17}$$

the averaged output state $\hat{\rho}_{\text{out}}$ of Bob becomes

$$\hat{\rho}_{\text{out}} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, \hat{A}(x, p)\hat{\rho}_{\text{in}}\hat{A}^\dagger(x, p). \tag{4.18}$$

Thus we have found the completely positive map representation, Eqs. (4.14) and (4.18), of the continuous variables teleportation.

The probability $P(x, p)$ that the measurement outcome (x, p) is obtained by Alice is nothing but the operational phase-space probability distribution. In fact, the probability $P(x, p)$ can be written in the following form:

$$P(x, p) = \frac{1}{2\pi}\, \text{Tr}[\hat{D}(x, p)\hat{\sigma}\,\hat{D}^\dagger(x, p)\hat{\rho}_{\text{in}}], \tag{4.19}$$

where the quantum ruler state $\hat{\sigma}$ is given by

$$\hat{\sigma} = 2\pi\,\hat{A}^\dagger\hat{A} = 2\pi \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, G'(x, y)|x\rangle\langle y|, \tag{4.20}$$

with

$$G'(x, y) = \frac{1}{\pi\sqrt{2\pi(e^{2r} + e^{-2r})}}\, \exp\left[-\frac{1}{8}(e^{2r} + e^{-2r})(x - y)^2 \right.$$
$$\left. - \frac{1}{2(e^{2r} + e^{-2r})}(x + y)^2\right]. \tag{4.21}$$

It is easy to see from Eqs. (4.7) and (4.15) that $\hat{\sigma} > 0$ and $\text{Tr}\hat{\sigma} = 1$. The probability $P(x, p)$ approaches to the uniform distribution as the squeezing parameter is sufficiently large.

We next show that the average value of the fidelity between the original and teleported quantum states is expressed in terms of the operational phase-space probability distribution. Let us introduce the operational probability distribution of the original quantum state $\hat{\rho}_{\rm in} = |\psi_{\rm in}\rangle\langle\psi_{\rm in}|$ in which the teleported quantum state $\hat{\rho}'_{\rm out}(x, p)$ before performing the unitary transformation $\hat{D}(x, p)$, that is, $\hat{\rho}'_{\rm out}(x, p) = \hat{D}^\dagger(x, p)\hat{\rho}_{\rm out}(x, p)\hat{D}$, is used as the quantum ruler state,

$$\mathcal{W}_{\rm tel}(r, k) = \frac{1}{2\pi}\,{\rm Tr}[\hat{D}(r, k)\hat{\rho}'_{\rm out}(x, p)\hat{D}^\dagger(r, k)\hat{\rho}_{\rm in}]. \tag{4.22}$$

Then it is found from Eq. (4.13) that the average value of the fidelity of the continuous variables teleportation is expressed as

$$F = 2\pi\int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty} dp\,\mathcal{W}_{\rm tel}(x, p)P(x, p), \tag{4.23}$$

where $P(x, p)$ is the probability that Alice obtains the measurement outcome (x, p). Therefore, it has been shown that the quantum teleportation of continuous variables is described by means of the operational phase-space probability distribution.

5. WIGNER–WEYL FUNCTION AND EXTENDED PHASE SPACE

5.1. Wigner and Weyl Functions

The Wigner function $W(x, p; \hat{\rho})$ of a quantum state $\hat{\rho}$ of a physical system is defined in the several ways. For example, we have the following expressions:

$$\begin{aligned} W(x, p; \hat{\rho}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dX\left\langle x + \frac{1}{2}X|\hat{\rho}|x - \frac{1}{2}X\right\rangle \exp(-ipX) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dP\left\langle p + \frac{1}{2}P|\hat{\rho}|p - \frac{1}{2}P\right\rangle \exp(iPx) \\ &= \frac{1}{\pi}\,{\rm Tr}[\hat{\rho}\hat{U}(x, p)], \end{aligned} \tag{5.1}$$

where $|x\rangle$ and $|p\rangle$ are the eigenstates of position and momentum operators, $\hat{x}$ and $\hat{p}$. In this equation, the unitary operator $\hat{U}(x, p)$ is given by

$$\hat{U}(x, p) = \hat{D}(x, p)\hat{U}_0\hat{D}^\dagger(x, p), \tag{5.2}$$

where $\hat{D}(x, p) = e^{i(p\hat{x} - x\hat{p})}$ is the displacement operator and $\hat{U}_0$ is the parity operator defined by

$$\hat{U}_0 = \exp\left[\frac{1}{2}i\pi(\hat{x}^2 + \hat{p}^2 - 1)\right], \tag{5.3}$$

which satisfies the relation $\hat{U}_0\hat{D}(x, p)\hat{U}_0^\dagger = \hat{D}(-x, -p) = \hat{D}^\dagger(x, p)$.

The Weyl function $\tilde{W}(X, P; \hat{\rho})$ of a quantum state $\hat{\rho}$ of a physical system is defined as

$$\begin{aligned}\tilde{W}(W, P; \hat{\rho}) &= \int_{-\infty}^{\infty} dx \left\langle x + \frac{1}{2}X|\hat{\rho}|x - \frac{1}{2}X\right\rangle \exp(-iPx)\\ &= \int_{-\infty}^{\infty} dp \left\langle p + \frac{1}{2}P|\hat{\rho}|p - \frac{1}{2}P\right\rangle \exp(ipX)\\ &= \frac{1}{\pi}\,\mathrm{Tr}[\hat{\rho}\hat{D}(x, p)],\end{aligned} \tag{5.4}$$

which is the Fourier transformation of the Wigner function $W(x, p; \hat{\rho})$,

$$\tilde{W}(X, P; \hat{\rho}) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, W(x, p; \hat{\rho}) \exp[-i(Px - pX)]. \tag{5.5}$$

The Weyl function satisfies the relations $\tilde{W}(0, 0; \hat{\rho}) = 1$ and $\tilde{W}(X, P; \hat{\rho}) = \tilde{W}^*(-X, -P; \hat{\rho})$.

Chountasis and Vourdas have introduced the following quantities (Chountasis and Vourdas, 1998b) to formulate the extended phase-space description of a quantum system,

$$\langle\langle F(x, p)\rangle\rangle_{(x,p)} = 2\pi \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, F(x, p)[W(x, p)]^2, \tag{5.6}$$

$$\langle\langle G(X, P)\rangle\rangle_{(X,P)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dP\, G(X, P)|\tilde{W}(X, P)|^2, \tag{5.7}$$

where $F(x, p)$ and $G(x, p)$ are some analytic functions of x and p. Using these quantities, they have discussed the uncertainty relations in the phase space. The uncertainty relations in the x–p–X–P phase space are given by

$$\delta X \delta p \geq \frac{1}{2}\,\mathrm{Tr}[\hat{\rho}^2], \qquad \delta x \delta P \geq \frac{1}{2}\,\mathrm{Tr}[\hat{\rho}^2], \tag{5.8}$$

where $\delta X(\delta P)$ and $\delta x(\delta p)$ are the fluctuations calculated respectively by $|\tilde{W}(X, P)|^2$ and $[W(x, p)]^2$. The properties of $|\tilde{W}(X, P)|^2$ and $[W(x, p)]^2$ have been investigated in detail by Chountasis and Vourdas (1998a). However, the physical meaning of the quantities calculated by $|\tilde{W}(X, P)|^2$ and $[W(x, p)]^2$ are not so clear. Therefore we will show that these quantities are related to those calculated by the operational phase-space probability distribution.

5.2. Wigner–Weyl Function in the Extended Phase Space

To investigate the relation of the extended phase-space quantities to the operational phase-space probability distribution, we introduce a Fourier transformation

$\tilde{\mathcal{W}}(r, k; \hat{\rho}, \hat{\sigma})$ of the operational phase-space probability distribution $\mathcal{W}(r, k; \hat{\rho}, \hat{\sigma})$,

$$\tilde{\mathcal{W}}(r, k; \hat{\rho}, \hat{\sigma}) = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, \mathcal{W}(r, k; \hat{\rho}, \hat{\sigma}) \exp[-i(Kr - Rk)] \tag{5.9}$$

$$\mathcal{W}(r, k; \hat{\rho}, \hat{\sigma}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, \tilde{\mathcal{W}}(R, K; \hat{\rho}, \hat{\sigma}) \exp[-i(Kr - Rk)]. \tag{5.10}$$

Substituting Eq. (2.3) into Eq. (5.9), we obtain

$$\begin{aligned} \tilde{\mathcal{W}}(R, K; \hat{\rho}, \hat{\sigma}) &= \tilde{W}(R, K; \hat{\rho}) \tilde{W}^*(R, K; \hat{\sigma}) \\ &= \tilde{W}(R, K; \hat{\rho}) \tilde{W}(-R, -K; \hat{\sigma}). \end{aligned} \tag{5.11}$$

For pure states $\hat{\rho} = |\psi\rangle\langle\psi|$ and $\hat{\sigma} = |\phi\rangle\langle\phi|$, the operational probability distribution and its Fourier transformation become

$$\mathcal{W}(r, k; \hat{\rho}, \hat{\sigma}) = \frac{1}{2\pi} |\langle\psi|\hat{D}(r, k)|\phi\rangle|^2, \tag{5.12}$$

$$\tilde{\mathcal{W}}(R, K; \hat{\rho}, \hat{\sigma}) = \langle\psi|\hat{D}^\dagger(R, K)|\psi\rangle\langle\phi|\hat{D}(R, K)|\phi\rangle, \tag{5.13}$$

which is the consequence of the following formula,

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk |\langle\psi|\hat{D}(r, k)|\phi\rangle|^2 \exp[-i(Kr - Rk)] \\ &= \langle\psi|\hat{D}^\dagger(R, K)|\psi\rangle\langle\phi|\hat{D}(R, K)|\phi\rangle. \end{aligned} \tag{5.14}$$

This formula indicates that the quantity $|\langle\psi|\hat{D}(r, k)|\psi\rangle|^2$ is invariant under the Fourier transformation. The characteristic function calculated by the operational phase-space probability distribution $\mathcal{W}(r, k; \hat{\rho}, \hat{\sigma})$ becomes

$$\begin{aligned} \langle \exp[-i(Kr - Rk)]\rangle_{\text{op}} &= \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, \mathcal{W}(r, k; \hat{\rho}, \hat{\sigma}) \exp[-i(Kr - Rk)] \\ &= \tilde{W}(R, K; \hat{\rho}) \tilde{W}^*(R, K; \hat{\sigma}). \end{aligned} \tag{5.15}$$

We have obtained the several relations between the operational phase-space probability distribution and the Weyl function. In the rest of this section, we set $\hat{o} = \hat{\sigma}$ and so we omit the statistical operators in the operational phase-space distribution, the Wigner function and the Weyl function.

We consider the relation between the operational phase-space probability distribution $\mathcal{W}(r, k)$ and the function $[W(x, p)]^2$ $|\tilde{W}(X, P)|^2$. It is easy to see from Eqs. (5.10), (5.11), and (5.15)

$$\langle \exp[-i(Kr - Rk)]\rangle_{\text{op}} = |\tilde{W}(R, K)|^2, \tag{5.16}$$

$$\langle\langle \exp[i(Pr - Xk)]\rangle\rangle_{(X,P)} = 2\pi \mathcal{W}(r, k). \tag{5.17}$$

On the other hands, the quantity $\langle\langle \exp[i(kx - rp)] \rangle\rangle_{(x,p)}$ is calculated to be

$$\langle\langle \exp[i(kx - rp)] \rangle\rangle_{(x,p)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dP\, \tilde{W}^*(X, P) \tilde{W}(X - r, P - k)$$
$$= \mathrm{Tr}\left[\hat{\rho} \hat{D}\left(\frac{1}{2}r, \frac{1}{2}k\right) \hat{\rho} \hat{D}\left(\frac{1}{2}r, \frac{1}{2}k\right)\right], \tag{5.18}$$

where we have used Eq. (5.11). These results show that the quantities calculated by the squared Wigner and Weyl functions are expressed in terms of the operational phase-space probability distributions. Thus the extended phase-space formulation proposed by Chountasis and Vourdas is implicitly related to the quantum measurement process of position and momentum.

6. CONCLUSION

We have shown that the operational phase-space probability distributions are useful tools for investigating quantum communication systems. It has been found that the quantum communication channels with Gaussian noise and the quantum teleportation of continuous variables are described by the operational phase-space probability distributions. Furthermore the relation of the operational phase-space probability distribution to the extended phase-space formalism proposed by Chountasis and Vourdas has also been discussed.

Before closing this paper, we consider the s-ordered quasiprobability distribution $\mathcal{F}(\alpha, s)$ of a quantum state $\hat{\rho}$ (Agarwal and Wolf, 1970a,b,c),

$$\mathcal{F}(\alpha, s) = \frac{1}{\pi} \mathrm{Tr}[\hat{\rho} \hat{\Delta}(\alpha, s)], \qquad \int d^2\alpha \mathcal{F}(\alpha, s) = 1, \tag{6.1}$$

where $0 \leq s \leq 1$ and the operator $\hat{\Delta}(\alpha, s)$ is given by

$$\hat{\Delta}(\alpha, s) = \frac{1}{\pi} \int d^2\beta \hat{D}(\beta) \exp\left[\left(s - \frac{1}{2}\right) |\beta|^2 - \beta\alpha^* + \beta^*\alpha\right]. \tag{6.2}$$

The quantum state $\hat{\rho}$ is expressed in terms of $\mathcal{F}(\alpha, s)$ and $\hat{\Delta}(\alpha, s)$,

$$\hat{\rho} = \int d^2\alpha \mathcal{F}(\alpha, s) \hat{\Delta}(\alpha, 1 - s). \tag{6.3}$$

It is easy to see that $\mathcal{F}(\alpha, 1)$, $\mathcal{F}(\alpha, 1/2)$, and $\mathcal{F}(\alpha, 0)$ become the Glauber–Sudarshan P-function, the Wigner function, and the Husimi Q-function. When we introduce an operator $\hat{\sigma}_s$,

$$\hat{\sigma}_s = (1 - e^{-\lambda}) \exp(-\lambda \hat{a}^\dagger \hat{a}), \tag{6.4}$$

where $e^{-\lambda} = -s/(1-s)$, the s-ordered quasiprobability distribution $\mathcal{F}(\alpha, s)$ can be expressed in the following form (Chaturvedi *et al.*, 1999):

$$\mathcal{F}(\alpha, s) = \frac{1}{\pi} \operatorname{Tr}[\hat{\rho} \hat{D}(\alpha) \hat{\sigma}_s \hat{D}^{\dagger}(\alpha)], \tag{6.5}$$

which is the same form as that of the operational phase-space probability distribution. However, since the operator $\hat{\sigma}_s$ does not represent any physical state, the s-ordered quasiprobability distribution $\mathcal{F}(\alpha, s)$ is not a probability distribution.

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